EUCLIDEAN AND NON-EUCLIDEAN GEOMETRY - WEEK 2

MICHAEL ALBANESE

Last week, we discussed the axioms of a field. These arose as the familiar properties of addition and multiplication of real numbers. We saw that many other things we were familiar with (such as subtraction, division, and the fact that division by zero is undefined) followed from these axioms. We also saw that when interpreting certain things appropriately (addition, multiplication, additive identity, multiplicative identity), there were many other examples that satisfied the axioms.

This week, we wish to have a similar discussion about planar geometry. One motivation for doing so is the observation that the geometry taught in high school is not actually the geometry of the world we live in. To better understand what geometry should mean, we want to have axioms or postulates that capture the notions we want.

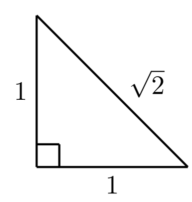
1. Euclid

In 300 BCE, the ancient Greek mathematician Euclid wrote 13 books called the Elements. It was the first systematic approach to geometry using what we recognise today as the axiomatic method. The first book contains five postulates, five common notions, 23 definitions, and 48 propositions (theorems). It is believed that Euclid may not be solely responsible for the authorship of these books. The mathematical content is not all originally due to Euclid, but builds what was known about geometry at the time from the same foundations. The Elements is considered to be the most influential textbook of all time, and up until the 20th century, students in geometry would learn from these books.

One drawback of Euclid's approach is that he does not refer to the real numbers. This is in part due to the lack of acceptance at the time of irrational numbers. Recall, a real number x is called *rational* if there are whole numbers a and b with $b \neq 0$ such that $x = \frac{a}{b}$. Note, by cancelling common factors, we can always arrange for a and b to have no common factors.

Example 1.1. $x = 2.5 = \dfrac{25}{10} = \dfrac{5 \times 5}{5 \times 2} = \dfrac{5}{2}$.

A real number which is not rational is called *irrational*. Such numbers arise frequently in geometry.



Theorem 1.2. *$\sqrt{2}$ is irrational.*

Proof. Suppose $\sqrt{2}$ is rational, then there are integers a and b with $b \neq 0$ which have no common factor such that $\sqrt{2} = \dfrac{a}{b}$.

$$\sqrt{2} = \frac{a}{b}$$

$$\begin{aligned} 2 &= \frac{a^2}{b^2} \\ 2b^2 &= a^2. \end{aligned}$$

Since $a^2 = 2b^2$, we see that a^2 is even and hence a is even. Writing $a = 2k$, we have

$$\begin{aligned} 2b^2 &= (2k)^2 \\ 2b^2 &= 4k^2 \\ b^2 &= 2k^2. \end{aligned}$$

Since $b^2 = 2k^2$, we see that b^2 is even and hence b is even. Therefore both a and b are divisible by 2, so 2 is a common factor. This contradicts our assumption about a and b. □

A geometric encapsulation of this result is that there is no right-angled isosceles triangle whose hypotenuse has length an integer.

We opt not to follow Euclid's postulates. There are lots of choices for the axioms/postulates of plane geometry since Euclid: Hilbert, Birkoff, etc. We choose to follow Lee's *Axiomatic Geometry*.

2. Axioms for Plane Geometry

P1 (The Set Postulate) Every line is a set of points, and there is a set of all points called the plane.

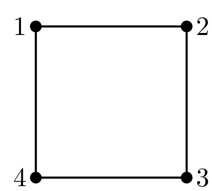
With only this postulate, we can't say anything as there may be no points in the plane at all! So we need the following to get started:

P2 (The Existence Postulate) There exists at least three distinct non-collinear points.

We say a collection of points are *collinear* if they all lie on the same line. Note, we now have some points, but we may have no lines. The following postulate ensures that we have lines as well:

P3 (The Unique Line Postulate) Given any two distinct points, there is a unique line that contains both of them.

Example 2.1. Consider the plane which consists of points $1, 2, 3, 4$ and lines $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\{4, 1\}$. This can be expressed in terms of the following diagram:



Note, the lines do not contain any other points aside from those that are indicated. It's easy to see that this satisfies P1 and P2, but it does not satisfy P3: there is no line containing 1 and 3, or a line containing 2 and 4.

Exercise 2.2 (Challenging). Consider the plane with 7 points 1, 2, 3, 4, 5, 6, 7 and 7 lines $\{1, 2, 3\}$, $\{3, 4, 5\}$, $\{5, 6, 1\}$, $\{1, 7, 4\}$, $\{3, 7, 6\}$, $\{5, 7, 2\}$, $\{2, 4, 6\}$. Draw a picture to illustrate this plane (known as the Fano plane).

Theorem 2.3. *Two distinct lines intersect at either no points or one point.*

Proof. Let ℓ_1 and ℓ_2 be two distinct lines and suppose they intersect at two or more points. Let A and B be two distinct intersection points. Note that A and B lie on both ℓ_1 and ℓ_2. This contradicts the uniqueness of a line through A and B given by P3. Therefore ℓ_1 and ℓ_2 intersect at either no points or one point. $\square$

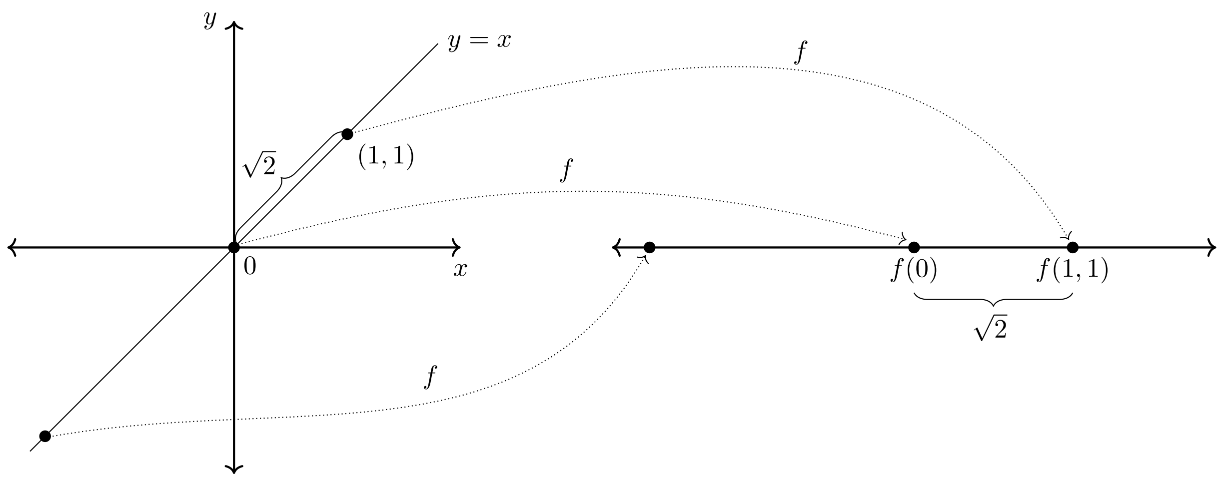
We say two lines which don't intersect are *parallel*. So the theorem above says that two distinct lines are either parallel or intersect in a single point.

P4 (The Distance Postulate) For every pair of points A and B, the distance from A to B is a non-negative real number determined by A and B.

P5 (The Ruler Postulate) For every line ℓ, there is a one-to-one correspondence between points on the line and the real numbers such that if A and B lie on the line, then the distance between A and B is the distance between the corresponding real numbers.

P5 essentially says that we can measure the distances in P4 (in particular, they aren't assigned randomly).

Consider the following diagram where the line in question is $y = x$ in the Cartesian plane and we measure distances in the usual way. We can view the copy of the real line on the right as a ruler for the line $y = x$ which allows us to measure distances along the line $y = x$. Here f denotes the one-to-one correspondence.



Note that the correspondence f is not unique. One may wonder if we can just map the point (x, x) on the line $y = x$ to the point x on the x-axis, but this doesn't work: the distance between $(0, 0)$ and $(1, 1)$ is $\sqrt{2}$, but the distance between 0 and 1 is $1 \neq \sqrt{2}$.

It is important to keep in mind that there is nothing in the postulates which says that distance has to be computed as Pythagoras' Theorem dictates. We will see later that we don't even have enough at this point to prove Pythagoras' Theorem.

P5 implies that every line has infinitely many points, so our previous examples with finitely many points do not satisfy P5.

Exercise 2.4. Show that there are infinitely many lines in the plane.

Note, none of the postulates mention the behaviour of the plane (aside from P1 which defines it).

P6 (The Plane Separation Postulate) For any line ℓ, the set of all points not on ℓ is the union of two disjoint subsets called the sides of ℓ. If A and B are distinct points on ℓ then A and B are on the same side of ℓ if and only if the line segment joining A and B does not intersect ℓ.

One can view this postulate as the one which ensures we're talking about a plane in the sense we understand it. For example, if one considers lines in three-dimensional spaces (with x, y, and z axes), then P1 through P5 are satisfied, but P6 is not.

Just as P4 and P5 introduced distances, P7 and P8 introduce angles.

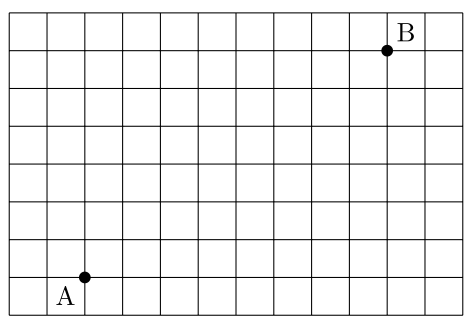
P7 (The Angle Measure Postulate) For every angle $\angle ab$, the measure of $\angle ab$ is a real number between 0 and 180 determined by $\angle ab$.

P8 (The Protractor Postulate) *This postulate is hard to state, but it essentially does for angles what the Ruler Postulate does for lengths.*

A *triangle* is the union of three line segments formed by three non-collinear points. Note, we can discuss line segments thanks to P5.

P9 (The SAS Postulate) If two triangles have a pair of sides of the same length, with the corresponding angle having the same measure, then the two triangles are congruent (all three sides and angles are the same).

Example 2.5 (Taxicab Geometry). Imagine a taxicab which has to drive in a city whose streets are aligned as a grid as below.



When measuring the distance from A to B, the length of the straight line which joins them isn't particularly helpful as the taxicab can't travel that way (when people describe the diagonal distance, it is usually followed by 'as the crow flies' since crows can travel that way). It is much more helpful to measure the distance between these points by the number of blocks the taxicab will have to drive. In this example, 8 blocks to the right and 6 up, for a total of 14. Note that the diagonal distance would be $\sqrt{8^2+6^2} = \sqrt{64+36} = \sqrt{100} = 10$.

Taxicab geometry is high school geometry where we replace the usual notion of distance with taxicab distance (in particular, angles are measured in the same way). Given points (a_1, a_2) and (b_1, b_2), the taxicab distance between them is $|a_2 - a_1| + |b_2 - b_1|$, where $|x|$ denotes the absolute value of x (equal to x if $x \geq 0$ and equal to $-x$ if $x < 0$).

Exercise 2.6. Consider two triangles in the Cartesian plane: one with vertices $(0,0)$, $(2,0)$, and $(0,2)$, and the other with vertices $(-1,0)$, $(1,0)$, and $(0,1)$. Find all the angles of each triangle and their sidelengths using the taxicab distance.

Note, the triangles in the above exercise have a pair of sides of the same length, with the corresponding angle having the same measure, but the two triangles are not congruent (the remaing sides have different lengths). That is, P9 is not satisfied in Taxicab Geometry (however, P1 through P8 are).

The above postulates are what is called *neutral geometry* or *absolute geometry.* In this geometry, we have the following result (which we will not prove).

Theorem 2.7. *For each line ℓ and each point P that does not lie of ℓ, there is a line that contains P and is parallel to ℓ.*

What this result does not tell us is how many such lines there are. The following postulate asserts there is only one:

P10 (The Parallel Postulate) For each line ℓ and each point P that does not lie on ℓ, there is a unique line that contains P and is parallel to ℓ.

The geometry one is taught in high school satisfies the above postulates (in particular, they are consistent, i.e., don't contradict each other). Note, this is not the version of the parallel postulate that appears in Euclid's Elements, but rather it is (almost) Playfair's axiom (which is equivalent).

One might wonder whether this postulate, which strengthens the previous theorem, is really necessary. That is, can we prove that the parallel line through P is unique just using P1 through P9? In an effort to answer this question, people looked for alternative statements which were equivalent to P10 and hoped to prove them instead.

Theorem 2.8. *If P1 though P9 are satisfied, then the following are equivalent:*

- *The Parallel Postulate (P10).*
- *If ℓ_1, ℓ_2, ℓ_3 are lines with ℓ_1 parallel to ℓ_2, and ℓ_2 parallel to ℓ_3, then ℓ_1 is parallel to ℓ_3.*
- *If ℓ_1 and ℓ_2 are parallel, and ℓ_3 intersects ℓ_1, then ℓ_3 intersects ℓ_2.*
- *The sum of angles in a triangle is $180°$.*
- *Pythagoras' Theorem*[1]*.*
- *There exists a rectangle.*
- *Given three non-collinear points, there is a circle that passes through them.*

There are many other statements which could be added to this list (about 50 of them).

This shows that if P1 through P9 are satisfied but P10 is not satisfied, then it is a form of geometry very different from what we are used to.

Question. (300 BCE) *If P1 through P9 are satisfied, is P10 necessarily satisfied? That is, can one use P1 through P9 to prove that for each line ℓ and each point P that does not lie on ℓ, there is a unique line that contains P and is parallel to ℓ?*

Answer. (2000 years later) *No! That is, you cannot prove the statement of P10 only using P1 through P9.*

We will see why next time.

University of Waterloo

Email address: m3albane@uwaterloo.ca

[1]In particular, we see that there is no reason why the distance between points from postulate 4 has to be given by the usual formula.